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A Fourier method for the fractional diffusion equation describing sub-diffusion ☆

Chang-Ming Chen [a], F. Liu [b,c,*], I. Turner [b], V. Anh [b]

[a] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[b] School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia
[c] School of Mathematical Sciences, South China University of Technology, Guangzhou 510640, China

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Abstract

In this paper, a fractional partial differential equation (FPDE) describing sub-diffusion is considered. An implicit difference approximation scheme (IDAS) for solving a FPDE is presented. We propose a Fourier method for analyzing the stability and convergence of the IDAS, derive the global accuracy of the IDAS, and discuss the solvability. Finally, numerical examples are given to compare with the exact solution for the order of convergence, and simulate the fractional dynamical systems.

Keywords: Fractional diffusion equation; Sub-diffusion; Implicit difference approximation; Fourier method; Stability; Convergence

1. Introduction

Fractional diffusion equations have attracted in recent years a considerable interest both in mathematics and in applications. These equations contain derivatives of fractional order in space, time or space–time [1]. They were used in modelling of many physical and chemical processes and in engineering [2–4]. Such evolution equations imply a fractional Fick's law for the flux that accounts for spatial and temporal non-locality [5]. Fractional calculus provides a powerful instrument for the description of memory and hereditary properties of substances [4]. Fractional-order differential equations have been the subject of worldwide attention by many research groups. In particular, the focus of Gorenflo, Mainardi and their co-authors' works on fractional calculus modelling (both deterministic and stochastic) and the derivation of fundamental solutions of the time, space and space–time fractional diffusion equations. They also presented discrete random walk models [6,7] and found that the fundamental solution can be interpreted as a probability

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* Corresponding author. Address: School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia. Tel.: +61 7 31385194; fax: +61 7 38642310.
E-mail addresses: fwliu@xmu.edu.cn, f.liu@qut.edu.au (F. Liu).

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density evolving in time of a self-similar stochastic process that can be viewed as a generalised diffusion process. Benson et al. [8,9] used a fractional advection–dispersion equation to simulate transport processes with heavy tails and demonstrated the equivalence between these heavy-tailed motions and transport equations that use fractional-order derivatives. Already in 1986, Wyss [10] considered the time fractional diffusion equation and gave the solution in closed form in terms of Fox functions. Then in 1989, Schneider and Wyss [11] considered the time fractional diffusion and wave equations, and the corresponding Green functions were obtained in closed form for arbitrary space dimensions in terms of Fox functions and their properties were exhibited. However, an explicit representation of the Green functions for the problem in a half-space was difficult to determine, except in the special cases $\alpha = 1$ (i.e., the first-order time derivative) with arbitrary n, or $n = 1$ with arbitrary α (i.e., the fractional-order time derivative). Huang and Liu [12] considered the time-fractional diffusion equations in an n-dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier–Laplace transform.

Fractional kinetic equations have proved particularly useful in the context of anomalous slow diffusion (sub-diffusion) [1]. The theoretical justification for the fractional diffusion equation, together with the abundance of physical and biological experiments demonstrating the prevalence of anomalous sub-diffusion, has led to an intensive effort in recent years to find accurate and stable methods of solution that are also straightforward to implement [13]. It has been suggested that the probability density function (pdf) $u(x, t)$ that describes anomalous sub-diffusive particles follows the fractional diffusion equation [1,13,14]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t), \quad t \geqslant 0, \tag{1}$$

where ${}_0D_t^{1-\gamma}u$ $(0 < \gamma \leqslant 1)$ denotes the Riemann–Liouville fractional derivative of order $1-\gamma$ of the function $u(x, t)$:

$${}_0D_t^{1-\gamma}u(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}\,\mathrm{d}\tau, \tag{2}$$

with $0 < \gamma < 1$. For $\gamma = 1$ one recovers the identity operator and for $\gamma = 0$ the ordinary first-order derivative.

Some numerical methods for solving the space or time, or time–space fractional partial differential equations have been proposed [15–24]. However, the stability and convergence of numerical methods for fractional partial differential equations are deserved further investigations.

In this paper, we consider the initial-boundary value problem of the fractional diffusion equation describing sub-diffusion (FDE-sub) [13,25]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t), \quad 0 < t \leqslant T, \; 0 < x < L, \tag{3}$$

$$u(0,t) = \varphi(t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

$$u(L,t) = \psi(t), \quad 0 \leqslant t \leqslant T, \tag{5}$$

$$u(x,0) = w(x), \quad 0 \leqslant x \leqslant L, \tag{6}$$

where $0 < \gamma \leqslant 1$; $f(x,t)$, $\varphi(t)$, $\psi(t)$ and $w(x)$ are sufficiently smooth functions.

Langlands and Henry [13] have investigated this problem. They proposed an implicit numerical scheme (L1 approximation), and discussed the accuracy and stability of this scheme. However, the global accuracy of the implicit numerical scheme has not been derived and it is apparent that the unconditional stability for all γ in the range $0 < \gamma \leqslant 1$ has not been established. The main purpose of this paper is to solve this problem via Fourier method.

The structure of the paper is as follows. In Section 2, we present an implicit difference approximation scheme. Sections 3 and 4 investigate the stability and convergence of the IDAS, respectively, using Fourier method. We prove that the IDAS is unconditionally stable for all γ in the range $0 < \gamma \leqslant 1$, derive the global accuracy of the IDAS, analyze the convergence of the IDAS , and discuss the solvability. Finally, some numerical examples are provided.

2. An implicit difference approximation scheme for FDE-sub

In this section, we first let

$$t_k = k\tau, \quad k = 0, 1, \ldots, N$$

and

$$x_j = jh, \quad j = 0, 1, \ldots, M,$$

respectively, where $\tau = T/N$ and $h = L/M$. For every $1-\gamma$, the Riemann–Liouville fractional derivative exists and coincides with the Grünwald–Letnikov fractional derivative. The relationship between the Riemann–Liouville and Grünwald–Letnikov definitions is also another consequence that is important for numerical approximation of FDE-sub; the formulation of applied problems; the manipulation with fractional derivatives; and the formulation of physically meaningful initial and boundary value problems. This allows the use of the Riemann–Liouville definition during problem formulation, and then the Grünwald–Letnikov definition for obtaining the numerical solution [15]. In proposing an approximation for FDE-sub, the key point is how to approximate the Riemann–Liouville fractional derivative. Using the relationship between Grünwald–Letnikov and Riemann–Liouville fractional derivatives [4], we have

$$_0D_t^{1-\gamma}f(t) = \lim_{\tau\to 0}\tau^{\gamma-1}\Delta_\tau^{1-\gamma}f(t) = \lim_{\tau\to 0}\tau^{\gamma-1}\sum_{l=0}^{[t/\tau]}(-1)^l\binom{1-\gamma}{l}f(t-l\tau).$$

Then the initial-boundary value problem of FDE-sub (3)–(6) can be approximated by the following implicit difference approximation scheme:

$$\frac{u_j^k - u_j^{k-1}}{\tau} = \frac{\tau^{\gamma-1}}{h^2}\sum_{l=0}^{k}\lambda_l\left(u_{j-1}^{k-l} - 2u_j^{k-l} + u_{j+1}^{k-l}\right) + f_j^k \quad (k = 1, 2, \ldots, N, j = 1, 2, \ldots, M-1), \tag{7}$$

$$u_0^k = \varphi(t_k) \quad (k = 0, 1, \ldots, N), \tag{8}$$

$$u_m^k = \psi(t_k) \quad (k = 0, 1, \ldots, N), \tag{9}$$

$$u_j^0 = w(x_j) \quad (j = 0, 1, \ldots, M) \tag{10}$$

where $\lambda_l = (-1)^l\binom{1-\gamma}{l}$, $l = 0, 1, \ldots, k$, and $f_j^k = f(x_j, t_k)$.

3. Stability of the implicit difference approximation scheme

In this section, we will analyze the stability of the IDAS (7)–(10). We first rewrite (7) as

$$u_j^k = u_j^{k-1} + \mu\sum_{l=0}^{k}\lambda_l\left(u_{j-1}^{k-l} - 2u_j^{k-l} + u_{j+1}^{k-l}\right) + \tau f_j^k \quad (k = 1, 2, \ldots, N, j = 1, 2, \ldots, M-1), \tag{11}$$

where $\mu = \frac{\tau^\gamma}{h^2}$.

Let U_j^k be the approximate solution of IDAS (7)–(10), and define

$$\rho_j^k = u_j^k - U_j^k \quad (k = 0, 1, \ldots, N, j = 1, 2, \ldots, M-1),$$

and

$$\rho^k = \left[\rho_1^k, \rho_2^k, \ldots, \rho_{M-1}^k\right]^{\mathrm{T}},$$

respectively. We obtain the following roundoff error equations:

$$\rho_j^k = \rho_j^{k-1} + \mu\sum_{l=0}^{k}\lambda_l\left(\rho_{j-1}^{k-l} - 2\rho_j^{k-l} + \rho_{j+1}^{K-l}\right) \quad (k = 1, 2, \ldots, N, j = 1, 2, \ldots, M-1). \tag{12}$$

We will analyze the stability of IDAS (7)–(10) by using Fourier method. Based on this, we define grid functions:

$$\rho^k(x) = \begin{cases} \rho_j^k, & \texttt{when}\ x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2}, \quad j = 1,2,\ldots,M-1, \\ 0, & \texttt{when}\ 0 \leqslant x \leqslant \frac{h}{2} \quad \texttt{or} \quad L - \frac{h}{2} < x \leqslant L, \end{cases} \qquad (k = 1,2,\ldots,N),$$

then $\rho^k(x)$ can be expanded in a Fourier series:

$$\rho^k(x) = \sum_{m=-\infty}^{\infty} d_k(m) e^{i2\pi mx/L} \quad (k = 1,2,\ldots,N),$$

where

$$d_k(m) = \frac{1}{L}\int_0^L \rho^k(x) e^{-i2\pi mx/L}\,dx.$$

Noticing, with the natural definition of the discrete 2-norm,

$$\begin{aligned} \|\rho^k\|_2 &= \left(\sum_{j=1}^{M-1} h|\rho_j^k|^2\right)^{\frac{1}{2}} = \left[\int_0^{\frac{h}{2}} |\rho^k(x)|^2 dx + \sum_{j=1}^{M-1}\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} |\rho^k(x)|^2 dx + \int_{L-\frac{h}{2}}^{L} |\rho^k(x)|^2 dx\right]^{\frac{1}{2}} \\ &= \left[\int_0^L |\rho^k(x)|^2 dx\right]^{\frac{1}{2}} \end{aligned} \tag{13}$$

and applying the Parseval equality:

$$\int_0^L |\rho^k(x)|^2 dx = \sum_{m=-\infty}^{\infty} |d_k(m)|^2,$$

we obtain

$$\|\rho^k\|_2^2 = \sum_{m=-\infty}^{\infty} |d_k(m)|^2. \tag{14}$$

Based on the above analysis, we can suppose that the solution of Eq. (12) has the following form:

$$\rho_j^k = d_k e^{i\sigma jh},$$

where $\sigma = 2\pi m/L$. Substituting the above expression into (12), we obtain

$$\left(1 + 4\sin^2\frac{\sigma h}{2}\right) d_k = \left(1 - 4\mu\lambda_1 \sin^2\frac{\sigma h}{2}\right) d_{k-1} - 4\mu \sin^2\frac{\sigma h}{2}\sum_{l=2}^{k}\lambda_l d_{k-l} \quad (k = 1,2,\ldots,N). \tag{15}$$

Lemma 1. *The coefficients λ_l $(l = 0,1,\ldots)$ satisfy*

(1) $\lambda_0 = 1,\ \lambda_1 = \gamma - 1,\ \lambda_l < 0,\ l = 1,2,\ldots$
(2) $\sum_{l=0}^{\infty}\lambda_l = 1$, *and* $\forall n \in N^+,\ -\sum_{l=1}^{n}\lambda_l < 1$.

Applying Lemma 1, Eq. (15) can be written as

$$d_k = \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}} d_{k-1} - \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}\lambda_l d_{k-l} \quad (k = 1,2,\ldots,N). \tag{16}$$

Proposition 1. *Supposing that d_k $(k = 1,2\ldots,N)$ be the solution of Eq. (16), we have*

$$|d_k| \leqslant |d_o| \quad (k = 1,2,\ldots,N).$$

Proof. We will use mathematical induction to complete the proof. For $k=1$, from Eq. (16) we have

$$d_1 = \frac{1+4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} d_0.$$

Noticing that $0<\gamma<1$, we obtain

$$|d_1| \leqslant |d_0|.$$

Supposing that

$$|d_n| \leqslant |d_0| \quad (n=1,2,\ldots,k-1),$$

applying Lemma 1, from Eq. (16), we have

$$\begin{aligned}|d_k| &\leqslant \frac{1+4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}}|d_{k-1}| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}|\lambda_l||d_{k-l}| \\ &\leqslant \left[\frac{1+4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}}\left(\sum_{l=1}^{k}|\lambda_l| - |\lambda_1|\right)\right]|d_0| \\ &= \left\{\frac{1+4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}}\left[-\sum_{l=1}^{k}\lambda_l - (1-\gamma)\right]\right\}|d_0| \\ &\leqslant \left\{\frac{1+4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}} + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1+4\mu\sin^2\frac{\sigma h}{2}}[1-(1-\gamma)]\right\}|d_0| = |d_0|.\end{aligned}$$

This completes the proof. □

Theorem 1. *The implicit difference approximation scheme (7)–(10) is unconditionally stable.*

Proof. Applying Proposition 1, and noticing (14), we obtain

$$\|\rho^k\|_2 \leqslant \|\rho^0\|_2, \quad k=1,2,\ldots,N,$$

which proves that IDAS (7)–(10) is unconditionally stable. □

4. Convergence of the implicit difference approximation scheme

In this section, we first introduce the following lemma.

Lemma 2. $\tau^{\gamma-1}\sum_{l=0}^{k}\lambda_l = \frac{1}{\Gamma(\gamma)} + 0(\tau)$.

Proof. Because

$${}_0D_t^{1-\gamma}g(t) = \tau^{\gamma-1}\sum_{l=0}^{[t/\tau]}\lambda_l g(t-l\tau) + 0(\tau),$$

then

$${}_0D_t^{1-\gamma}g(t)|_{t=t_k} = \tau^{\gamma-1}\sum_{l=0}^{k}\lambda_l g(t_k-l\tau) + 0(\tau). \tag{17}$$

Taking $g(t)=1$ and $t_\mathrm{k}=1$ in (17), we have

$${}_0D_t^{1-\gamma}(1)|_{t=1} = \tau^{\gamma-1}\sum_{l=0}^{k}\lambda_l + 0(\tau).$$

Therefore,

$$\tau^{\gamma-1}\sum_{l=0}^{k}\lambda_l = {}_0D_t^{1-\gamma}(1)|_{t=1} + 0(\tau) = \frac{1}{\Gamma(\gamma)} + 0(\tau).$$

This completes the proof. □

We now define

$$R_j^k = \frac{u(x_j,t_k)-u(x_j,t_{k-1})}{\tau} - \frac{\tau^{\gamma-1}}{h^2}\sum_{l=0}^{k}\lambda_l[u(x_{j-1},t_{k-l}) - 2u(x_j,t_{k-l}) + u(x_{j+1},t_{k-l})]$$
$$(k=1,2,\ldots,N, j=1,2,\ldots,M-1). \tag{18}$$

Applying Lemma 2 and (17), we have

$$\frac{\tau^{\gamma-1}}{h^2}\sum_{l=0}^{k}\lambda_l[u(x_{j-1},t_{k-l}) - 2u(x_j,t_{k-l}) + u(x_{j+1},t_{k-l})] = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x_j,t_k)}{\partial x^2}\right] + \mathrm{O}(\tau) + \mathrm{O}(h^2)$$

On the other hand,

$$\frac{u(x_j,t_k)-u(x_j,t_{k-1})}{\tau} = \frac{\partial u(x_j,t_k)}{\partial t} + \mathrm{O}(\tau).$$

Consequently,

$$R_j^k = \mathrm{O}(\tau+h^2) \quad (k=1,2,\ldots,N, j=1,2,\ldots,M-1).$$

Therefore, there is a positive constant c_1, such that

$$|R_j^k| \leqslant c_1(\tau+h^2), \quad k=1,2,\ldots,N \quad (j=1,2,\ldots,M-1). \tag{19}$$

Let

$$e_j^k = u(x_j,t_k) - u_j^k \quad (k=1,2,\ldots,N,\ i=1,2,\ldots,M-1)$$

and

$$e^k = \left[e_1^k, e_2^k, \ldots, e_{M-1}^k\right]^{\mathrm{T}}, \quad R^k = \left[R_1^k, R_2^k, \ldots, R_{M-1}^k\right]^{\mathrm{T}},$$

respectively. From (18), we have

$$u(x_j,t_k) = u(x_j,t_{k-1}) + \frac{\tau^{\gamma}}{h^2}\sum_{l=0}^{k}\lambda_l[u(x_{j-1},t_{k-l}) - 2u(x_j,t_{k-l}) + u(x_{j+1},t_{k-l})] + \tau f(x_j,t_k) + \tau R_j^k,$$
$$k=1,2,\ldots,N, \quad j=1,2,\ldots,M-1.$$

Subtracting the above equation from Eq. (11), we obtain

$$e_j^k = e_j^{k-1} + \mu\sum_{l=0}^{k}\lambda_l\left(e_{j-1}^{k-l} - 2e_j^{k-l} + e_{j+1}^{k-l}\right) + \tau R_j^k \quad (k=1,2\ldots,N, j=1,2,\ldots,M-1). \tag{20}$$

We now analyze the convergence of IDAS (7)–(10) by using Fourier method. Using the same idea to the stability analysis in Section 3, we first define grid functions:

$$e^k(x) = \begin{cases} e_j^k, & \texttt{when}\ x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2},\ j=1,2,\ldots,M-1, \\ 0, & \texttt{when}\ 0 \leqslant x \leqslant \frac{h}{2} \quad \texttt{or} \quad L-\frac{h}{2} < x \leqslant L, \end{cases} \quad (k=0,1,\ldots,N)$$

and

$$R^k(x) = \begin{cases} R_j^k, & \text{when } x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2},\ \ j = 1, 2, \ldots, M-1, \\ 0, & \text{when } 0 \leqslant x \leqslant \frac{h}{2} \quad \text{or} \quad L - \frac{h}{2} < x \leqslant L, \end{cases} \qquad (k = 1, 2, \ldots, N),$$

respectively. Then $e^k(x)$ and $R^k(x)$ have Fourier series expansions, respectively:

$$e^k(x) = \sum_{m=-\infty}^{\infty} \xi_k(m) \mathrm{e}^{\mathrm{i}2\pi mx/L} \quad (k = 0, 1, \ldots, N)$$

and

$$R^k(x) = \sum_{m=-\infty}^{\infty} \eta_k(m) \mathrm{e}^{\mathrm{i}2\pi mx/L} \quad (k = 1, 2, \ldots, N)$$

where

$$\xi_k(m) = \frac{1}{L} \int_0^L e^k(x) \mathrm{e}^{-\mathrm{i}2\pi mx/L} \mathrm{d}x,$$

and

$$\eta_k(m) = \frac{1}{L} \int_0^L R^k(x) \mathrm{e}^{-\mathrm{i}2\pi mx/L} \mathrm{d}x.$$

Similarly, we also have

$$\|e^k\|_2^2 = \left(\sum_{j=1}^{M-1} h |e_j^k|^2 \right)^{\frac{1}{2}} = \sum_{m=-\infty}^{\infty} |\xi_k(m)|^2 \quad (k = 0, 1, \ldots, N) \tag{21}$$

and

$$\|R^k\|_2^2 = \left(\sum_{j=1}^{M-1} h |R_j^k|^2 \right)^{\frac{1}{2}} = \sum_{m=-\infty}^{\infty} |\eta_k(m)|^2 \quad (k = 1, 2, \ldots, N), \tag{22}$$

respectively. Based on the above analysis, we can suppose that

$$e_j^k = \xi_k \mathrm{e}^{\mathrm{i}\sigma jh}$$

and

$$R_j^k = \eta_k \mathrm{e}^{\mathrm{i}\sigma jh},$$

respectively. Substituting the above expressions into (20), we obtain

$$\left(1 + 4\mu \sin^2 \frac{\sigma h}{2}\right) \xi_k = \left(1 - 4\mu\lambda_1 \sin^2 \frac{\sigma h}{2}\right) \xi_{k-1} - 4\mu \sin^2 \frac{\sigma h}{2} \sum_{l=2}^{k} \lambda_l \xi_{k-l} + \tau \eta_k \quad (k = 1, 2, \ldots, N). \tag{23}$$

Applying Lemma 1, Eq. (23) can be written as

$$\xi_k = \frac{1 + 4\mu(1-\gamma) \sin^2 \frac{\sigma h}{2}}{1 + 4\mu \sin^2 \frac{\sigma h}{2}} \xi_{k-1} - \frac{4\mu \sin^2 \frac{\sigma h}{2}}{1 + 4\mu \sin^2 \frac{\sigma h}{2}} \sum_{l=2}^{k} \lambda_l \xi_{k-l} + \frac{\tau \eta_k}{1 + 4\mu \sin^2 \frac{\sigma h}{2}} \quad (k = 1, 2, \ldots, N). \tag{24}$$

Proposition 2. *Suppose the $\xi_k (k = 1, 2, \ldots, N)$ be the solution of Eq. (24), then there is a positive constant c_2 such that*

$$|\xi_k| \leqslant k\tau c_2 |\eta_1|, \quad k = 1, 2, \ldots, N.$$

Proof. First, noticing that $e^0 = 0$, we have

$$\xi_0 \equiv \xi_0(m) = 0.$$

In addition, from (19) and the left-hand side of (22), we have

$$\|R^k\|_2 \leqslant c_1\sqrt{L}(\tau + h^2), \quad k = 1, 2, \ldots, N. \tag{25}$$

Again, based on the convergence of the series in the right-hand side of (22), then there is a positive constant c_2 such that

$$|\eta_k| \equiv |\eta_k(m)| \leqslant c_2|\eta_1| \equiv c_2|\eta_1(m)|, \quad k = 1, 2, \ldots, N. \tag{26}$$

We will complete the proof using mathematical induction. For $k = 1$, from (24), we have

$$\xi_1 = \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\xi_0 + \frac{\tau\eta_1}{1 + 4\mu\sin^2\frac{\sigma h}{2}} = \frac{\tau\eta_1}{1 + 4\mu\sin^2\frac{\sigma h}{2}}.$$

From (26), we obtain

$$|\xi_1| \leqslant \tau|\eta_1| \leqslant c_2\tau|\eta_1|.$$

Suppose that

$$|\xi_n| \leqslant c_2 n\tau|\eta_1|, \quad n = 1, 2, \ldots, k-1.$$

Applying Lemma 1, and noticing that $0 < \gamma < 1$ and (26), from (24), we have

$$|\xi_k| \leqslant \frac{1 + 4\mu(1-\gamma)\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}|\xi_{k-1}| + \frac{4\mu\sin^2\frac{\sigma h}{2}}{1 + 4\mu\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}|\lambda_l||\xi_{k-l}| + \frac{\tau|\eta_k|}{1 + 4\mu\sin^2\frac{\sigma h}{2}} \leqslant c_2 k\tau|\eta_1|.$$

This completes the proof. □

Theorem 2. *The implicit difference approximation scheme (7)–(10) is L_2-convergent, and the order of convergence is* $O(\tau + h^2)$.

Proof . Applying Proposition 2 and (25), and noticing (21) and (22), we obtain

$$\|e^k\|_2 \leqslant c_2 k\tau\|R^1\|_2 \leqslant c_1c_2k\tau\sqrt{L}(\tau + h^2).$$

Because $k\tau \leqslant T$, we have

$$\|e^k\|_2 < c(\tau + h^2),$$

where $c = c_1c_2T\sqrt{L}$. This completes the proof. □

5. The solvability of the implicit difference approximation scheme

We let

$$u^0 = [w(x_1), w(x_2), \ldots, w(x_{M-1})]^{\mathrm{T}}$$

and

$$u^k = \left[u_1^k, u_2^k, \ldots, u_{M-1}^k\right]^{\mathrm{T}}, \quad k = 1, 2, \ldots, N$$

respectively, then the implicit difference approximation scheme (7)–(10) can be written in matrix form:

$$Au^k = \sum_{i=0}^{k-1} B_i u^i + F, \quad k = 1, 2, \ldots, N, \tag{27}$$

where

$$A=\begin{bmatrix}1+2\mu & -\mu & & & \\ -\mu & 1+2\mu & -\mu & & \\ & \ddots & \ddots & \ddots & \\ & & -\mu & 1+2\mu & -\mu \\ & & & -\mu & 1+2\mu\end{bmatrix},$$

$$B_i=\mu\lambda_{k-i}\begin{bmatrix}-2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2\end{bmatrix},\quad i=0,1,\ldots,k-2,$$

$$B_{k-1}=\begin{bmatrix}1-2\mu\lambda_1 & \mu\lambda_1 & & & \\ \mu\lambda_1 & 1-2\mu\lambda_1 & \mu\lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu\lambda_1 & 1-2\mu\lambda_1 & -\mu\lambda_1 \\ & & & \mu\lambda_1 & 1-2\mu\lambda_1\end{bmatrix},$$

$$F=\begin{bmatrix}\mu\sum_{i=0}^{k}\lambda_{k-i}\varphi(t_i)+\tau f(x_1,t_k) \\ \tau f(x_2,t_k) \\ \vdots \\ \tau f(x_{M-2},t_k) \\ \mu\sum_{i=0}^{k}\lambda_{k-i}\psi(t_i)+\tau f(x_{M-1},t_k)\end{bmatrix}.$$

Theorem 3. *The difference equation (27) is uniquely solvable.*

Proof . Because $\mu > 0$, then the coefficient matrix of the difference equation (27) is a strictly diagonally dominant matrix. Therefore, A is a nonsingular matrix; this proves Theorem 3. □

6. Numerical examples

In this section, some numerical examples are presented which confirm our theoretical results.

Example 1. Fractional diffusion equation describing sub-diffusion with a non-homogeneous term:

$$\frac{\partial u(x,t)}{\partial t}={}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]+e^x\left[(1+\gamma)t^\gamma-\frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}t^{2\gamma}\right],\quad 0<t\leqslant 1,\ 0<x<1, \tag{28}$$

$$u(0,t)=t^{1+\gamma},\quad 0\leqslant t\leqslant 1, \tag{29}$$

$$u(1,t)=et^{1+\gamma},\quad 0\leqslant t\leqslant 1, \tag{30}$$

$$u(x,0)=0,\quad 0\leqslant x\leqslant 1. \tag{31}$$

The exact solution of the problems (28)–(31) is

$$u(x,t)=e^x t^{1+\gamma}.$$

The maximum error of the exact solution and IDAS is defined as follows:

$$E_\infty = \max_{0 \leqslant k \leqslant N} \max_{0 \leqslant j \leqslant M} \left\{ |u_j^k - u(x_j, t_k)| \right\}.$$

A comparison of the maximum errors of the problems (28)–(31) at all mesh points for different γ between the L_1-approximation and IDAS is listed in Tables 1 and 2. From Tables 1 and 2, it can be seen that our IDAS is slightly more accurate than the L_1-approximation. Tables 3 and 4 show the maximum errors of the problems (28)–(31) at all mesh points for different γ using $\tau = h = \frac{1}{8}$ and $\tau = h = \frac{1}{32}$, respectively. From Tables 1–4, it can be seen that the IDAS is unconditionally stable and convergent with order $O(\tau + h^2)$, which conforms with our theoretical analysis.

Example 2. Fractional diffusion equation describing sub-diffusion with a homogeneous term:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right], \quad 0 < t, \ 0 < x < 2, \tag{32}$$

$$u(0,t) = 0, \quad 0 \leqslant t, \tag{33}$$

$$u(2,t) = 0, \quad 0 \leqslant t \leqslant 1, \tag{34}$$

$$u(x,0) = w(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{4-2x}{3}, & \frac{1}{2} \leqslant x \leqslant 2. \end{cases} \tag{35}$$

Table 1
The maximum error ($\tau = \frac{1}{64}$, $h = \frac{1}{8}$)

γ	E_∞ (IDAS)	E_∞ (L_1-approximation)
0.4	0.9774769E − 03	0.1812220E − 02
0.5	0.1314691E − 02	0.2103329E − 02
0.6	0.1640956E − 02	0.2363563E − 02

Table 2
The maximum error ($\tau = \frac{1}{1024}$, $h = \frac{1}{32}$)

γ	E_∞ (IDAS)	E_∞ (L_1-approximation)
0.4	0.1204014E − 03	0.2110046E − 03
0.5	0.9040628E − 04	0.1107985E − 03
0.6	0.2180338E − 03	0.2259971E − 03

Table 3
The maximum error ($\tau = h = \frac{1}{8}$)

γ	E_∞ (IDAS)
0.4	0.5480236E − 02
0.5	0.8357003E − 02
0.6	0.1132181E − 01

Table 4
The maximum error ($\tau = h = \frac{1}{32}$)

γ	E_∞ (IDAS)
0.4	0.1792436E − 02
0.5	0.2493483E − 02
0.6	0.3179647E − 02

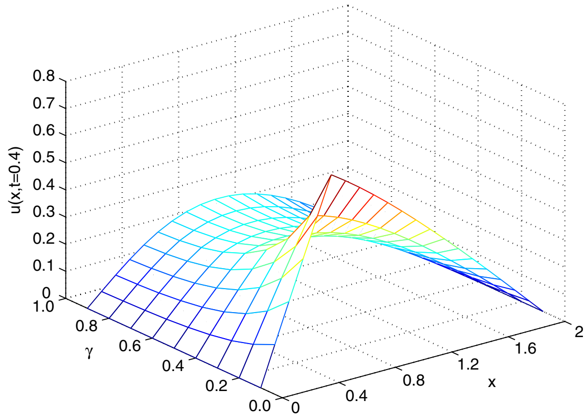

Fig. 1. The numerical solution of problem (32)–(35) when $t = 0.4$ for various γ.

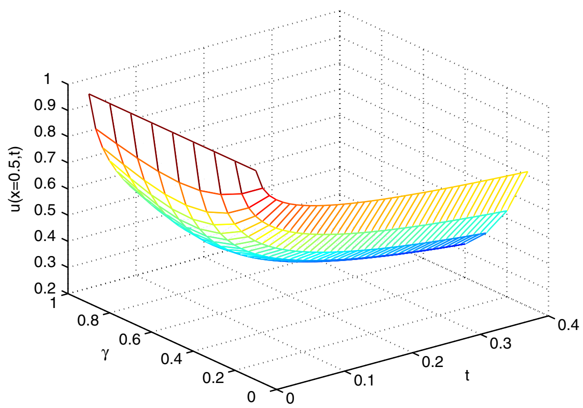

Fig. 2. The numerical solution of problem (32)–(35) when $x = \frac{1}{2}$ for various γ.

The function $w(x)$ represents the temperature distribution in a bar generated by a point heat source kept at the point $x = \frac{1}{2}$ for sufficiently long time.

Figs. 1 and 2 compare the response of the diffusion system for different real number $0 < \gamma < 1$ at $t = 0.4$ and different x, and at $x = 0.5$ and different t, respectively. In the example, we take $\tau = 0.01, h = 0.1$. From Figs. 1 and 2, it is seen that IDAS can be applied to simulate fractional dynamical systems.

7. Conclusion

In this paper, we presented an implicit difference approximation scheme for solving a fractional diffusion equation describing sub-diffusion. Fourier method has been has been used to successfully analyze the stability and the convergence of the IDAS. This technique can also be extended to analyze other fractional partial differential equations.

References

[1] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[2] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wein and New York, 1997, pp. 223–276.
[3] F. Mainardi, Fractional relaxation–oscillation and fractional diffusion-wave phenomena, Chaos Soliton Fract. 7 (9) (1996) 1461–1477.

[4] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[5] R. Gorenflo, F. Mainardi, D. Moretti, P. Paradisi, Time fractional diffusion: a discrete random walk approach, Nonlinear Dyn. 29 (2002) 129–143.
[6] R. Gorenflo, F. Mainardi, Random walk models for space fractional diffusion processes, Fractional Calc. Appl. Anal. 1 (1998) 167–191.
[7] F. Mainardi, Yu. Luchko, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation, Fractional Calc. Appl. Anal. 4 (2) (2001) 153–192.
[8] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection–dispersion equation, Water Resour. Res. 36 (6) (2000) 1403–1412.
[9] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Levy motion, Water Resour. Res. 36 (6) (2000) 1413–1423.
[10] W. Wyss, The fractional diffusion equation, J. Math. Phys. 27 (1986) 2782–2785.
[11] W.R. Schneider, W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 134–144.
[12] F. Huang, F. Liu, The time fractional diffusion and advection–dispersion equation, ANZIAM J. 46 (2005) 317–330.
[13] T.A.M. Langlands, B.I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, J. Comp. Phys. 205 (2005) 719–736.
[14] S.B. Yuste, L. Acedo, An explicit finite difference method and a new Von neumman-type stability analysis for fractional diffusion equations, SIAM J. Numer. Anal. 42 (5) (2005) 1862–1874.
[15] F. Liu, V. Anh, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, J. Comp. Appl. Math. 166 (2004) 209–219.
[16] F. Liu, V. Anh, I. Turner, P. Zhuang, Numerical simulation for solute transport in fractal porous media, ANZIAM J. 45 (E) (2004) 461–473.
[17] M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection–dispersion flow equations, J. Comp. Appl. Math. 172 (2004) 65–77.
[18] S. Shen, F. Liu, Error analysis of an explicit finite difference approximation for the space fractional diffusion, ANZIAM J. 46 (E) (2005) 871–887.
[19] F. Liu, S. Shen, V. Anh, I. Turner, Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation, ANZIAM J. 46 (E) (2005) 488–504.
[20] J.P. Roop, Computational aspects of FEM approximation of fractional advection dispersion equations on boundary domains in R^2, J. Comput. Appl. Math. 193 (1) (2006) 243–268.
[21] Q. Liu, F. Liu, I. Turner, V. Anh, Approximation of the Levy–Feller advection–dispersion process by random walk and finite difference method, J. Phys. Comp. 222 (2007) 57–70.
[22] P. Zhuang, F. Liu, Implicit difference approximation for the time fractional diffusion equation, J. Appl. Math. Comput. 22 (3) (2006) 87–99.
[23] F. Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage , Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation, Appl. Math. Comput., (2007), in press.
[24] H. Zhang, F. Liu, V. Anh, Numerical approximation of Levy–Feller diffusion equation and its probability interpretation, J. Comput. Appl. Math. 206 (2007) 1098–1115.
[25] F. So, K.L. Liu, A study of the subdiffusive fractional Fokker–Plank equation of bistable systems, Physica A 331 (2004) 378–390.